Wealth accumulation with random redistribution

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We study the wealth distribution in random multiplicative processes with random redistribution. The equilibrium distribution can be extended to the negative wealth. The extreme wealths follow power law distributions and the same exponent is found for both the large wealths and the large debts. We propose a mean-field model to emphasize the fluctuations in the thermodynamic limit. The exact solution can be obtained analytically.

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I. INTRODUCTION

With recent interactions between economists and physicists, the analogy between economics system and physics system starts to emerge [1]. As fluctuations are prominent in economy, statistical physics provides a theoretical framework to understand the empirically observed features. One of the interesting phenomena is the emergence of power law distributions, which imply large fluctuations. The distribution of large wealths following a power law has been noticed for over a century [2]. The same power law can be applied to different countries or social organizations, even in the ancient Egyptian society [3–5]. However, the theoretical basis to understand the existence of power law distributions and the value of its exponents is still desired. Recently, various models have been proposed to reproduce the empirical scaling [6–9]. A coherent understanding is expected to be well underway.

Though the power law distribution has been the focus of recent research interests, it actually accounts only a small portion of the overall distribution. The large wealths subjected to the power law behavior are estimated to be no more than a few percents. For the majority of the so called middle class, a much more steepened distribution is observed. The empirical data can be fairly fitted by either the exponential distribution [11] or the log-normal distribution [12]. As to the distribution of the small wealths, the situations are much more confused. In the framework of the log-normal distributions, the weightings drop to zero sharply as the wealth decreases; while in that of the exponential distributions, as the wealth decreases, the weightings monotonically increase and maximize at the zero wealth. The data seem to suggest a situation in between these two, i.e., the distribution drops as the wealth decreases, but not as fast as the log-normal distribution suggested. And a small yet finite weighting should be assigned to the zero wealth, which makes it natural to be extent to the negative wealths. In previous studies, the wealth is often assumed to be a positive definite quantity and different mechanisms are resorted to explain the features in different parts of the distribution. It would be interesting to have a simple model able to present these different features from the same mechanism.

In this paper, we follow the attempt of Ref. [9] which provides a mean-field model to describe the power law distribution of large wealths. The model assumes a random multiplicative process for the fluctuations. Both multiplicative and additive noises are included. However, the multiplicative fluctuations are random; while the additive ones are deterministic. We go beyond the mean-field assumption and consider further noise in the additive term. The model will be described in the following section. Both numerical and analytical results are presented. In Sec. III, a new mean-field model is proposed to preserve the fluctuations in the thermodynamic limit. The analytical results exactly reproduce the numerical data. The concluding remarks are presented in the last section.

II. RANDOM REDISTRIBUTION

Consider an economy consisting of N individuals. The wealth of each person is affected by two factors: growth and redistribution of the total wealth. We assume that the wealth w_i of the ith person is described by the following stochastic differential equation [9]:

$$\frac{dw_i}{dt} = \eta_i w_i + \frac{1}{N}\sum_j (J_{ij}w_j - J_{ji}w_i), \tag{1}$$

where the index $i=1,2,\cdots N$. The first term prescribes a multiplicative noise for the spontaneous growth or decline. The random variable η_i is Gaussian with mean m and variance σ_m. The second term describes a trade between person i and person j, and thus accounts for the wealth redistribution. It can be taken as that an amount of money $(J_{ij}\ w_j)$ has been earned by the person i, or $(J_{ji}\ w_i)$ been spent. The conservation of total money in each trade is ensured. It is also reasonable to assume that the two persons trade with each other at the same rate, i.e., $J_{ij}=J_{ji}$. The random variable J_{ij} is also Gaussian with mean J and variance σ_J. The model can be labeled as a random multiplicative process with random redistribution. The interactions are completely specified by four parameters (m,σ_m,J,σ_J). The spontaneous growth is controlled by m and σ_m; the redistribution is controlled by J and σ_J.

When the effect of redistribution is totally neglected, i.e., $J=0$ and $\sigma_J=0$, the wealth distribution has the following asymptotic probability density [10]:

$$P(\ln w,t) = \frac{1}{\sqrt{2\pi\sigma_m^2 t}}\exp\left[-\frac{(\ln w - mt)^2}{2\sigma_m^2 t}\right]. \tag{2}$$

The average wealth $\overline{w}$ has a time dependence as following:

$$\overline{w}(t) = \overline{w}(0)\exp\left[\left(m + \frac{\sigma_m^2}{2}\right)t\right]. \tag{3}$$

When a deterministic redistribution is considered, i.e., $J>0$ and $\sigma_J=0$, the wealth distribution has the following stationary solution [9]:

$$P(a) = \frac{(\alpha-1)^{\alpha}}{\Gamma(\alpha)}\frac{\exp\left[-\frac{\alpha-1}{a}\right]}{a^{1+\alpha}}, \tag{4}$$

where $a=w/\overline{w}$ is the normalized wealth and the exponent $\alpha=1+2J/\sigma_m^2$. The average wealth is still described by Eq. (3). The deterministic redistribution provides a mechanism to reproduce the Pareto power law tail for the large wealths [2],

$$P(a \gg 1) \propto \frac{1}{a^{1+\alpha}}. \tag{5}$$

The exponent α increases with the increase of J, and also with the decrease of σ_m.

Next, we consider the random noise in redistribution. In this work, we adopt a discrete version of the model as the following update rule:

$$w_i \to w_i \exp(\eta_i) + \frac{1}{N}\sum_j (J_{ij} w_j - J_{ji} w_i). \tag{6}$$

In the numerical simulations, we start with a uniform configuration, i.e., initially all the personal wealths are the same. As the scale is irrelevant, we simply assign $w_i=1$ at $t=0$. As time evolves, a stationary distribution emerges asymptotically. The typical results are shown in Fig. 1. For large wealths, the same power law distribution shown in Eq. (5) is preserved. The exponent is determined by parameters σ_m and J, and independent of σ_J and m. As the fluctuation σ_J increases, the power law distribution keeps the same exponent, but enhances its normalization, i.e., the large wealths occupy much more weighting of the distribution. It is interesting to observe that a nonvanish weighting is constituted by the negative wealth $w<0$. Without noise, $\sigma_J=0$, the equilibrium distribution is confined to the range $a>0$, i.e., all the personal wealths are positive definite. With fluctuation, $\sigma_J>0$, the wealth distribution is extended to $a<0$, i.e., some people become in debt. With randomized trading rates, the large wealths emerge more swiftly and prominently. However, the fluctuations also cause some people to lose all their wealth. In this work, we do not set a lower bound to the personal wealth. For those people in debt, they can still participate in the trading. And their debts are also subjected to the spontaneous fluctuations.

We also notice that such a surprising effect is a finite size effect. In this model, the fluctuations diminish as the number of participants (N) increases. In the thermodynamic limit $N \to\infty$, the fluctuations are sufficiently suppressed and the positive-definite distribution in Eq. (4) is restored.

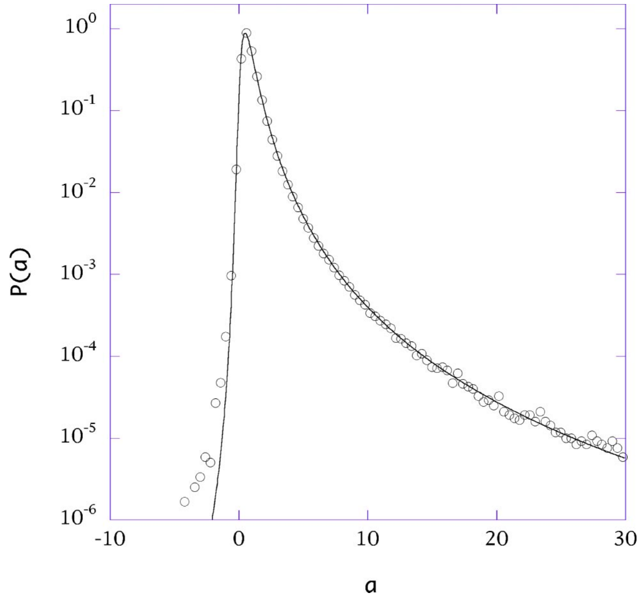

FIG. 1. Wealth distribution in the model of Eq. (1). The parameters are $(m,\sigma_m,J,\sigma_J)=(-0.005,0.1,0.01,1)$ and $N=1000$. As we do not emphasize the growth of total wealth, a setting of $m=-\sigma_m^2/2$ is chosen, see Eq. (3). The numerical data are averaged over 3000 simulations at $t=1000$. The analytical results of Eq. (10) are shown by the solid line.

In terms of the normalized wealth $a_i=w_i/\overline{w}$ and Eq. (3), the model can be reexpressed as

$$\frac{da_i}{dt} = \left(\eta_i - m - \frac{\sigma_m^2}{2}\right)a_i + \frac{1}{N}\sum_j J_{ij}(a_j - a_i). \tag{7}$$

As the variables a_i are all coupled together by the random trading, the corresponding Fokker-Planck equation cannot be obtained straightforwardly. However, we observe that the right hand side of Eq. (7) can be quasiseparated into a deterministic part $[-(\sigma_m^2/2)a_i+J(1-a_i)]$ and a stochastic part $[(\eta_i-m)a_i-J+(\sum_j J_{ij}a_j)/N]$. And the variance of the stochastic noise can be further approximated by $[\sigma_m^2 a_i^2+x/N]$, where $x=(J^2+\sigma_J^2)\langle a^2\rangle-J^2$. Then the evolution of the probability density $P(a,t)$ can be obtained as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial a}\left\{\left[\frac{\sigma_m^2}{2}a - J(1-a)\right]P\right\} + \frac{\sigma_m^2}{2}\frac{\partial}{\partial a}\left[a\frac{\partial}{\partial a}(aP)\right] + \frac{x}{2N}\frac{\partial^2 P}{\partial a^2}. \tag{8}$$

With the equilibrium condition, we have

$$\frac{dP}{da} = -2\left[\frac{(\sigma_m^2+J)a - J}{\sigma_m^2 a^2 + \frac{x}{N}}\right]P. \tag{9}$$

The solution of the equilibrium distribution becomes

$$P(a) \propto \frac{\exp\left[\frac{2J}{\sigma_m}\sqrt{\frac{N}{x}}\tan^{-1}\left(\sigma_m\sqrt{\frac{N}{x}}\,a\right)\right]}{\left(\sigma_m^2 a^2 + \frac{x}{N}\right)^{1+(J/\sigma_m^2)}}. \qquad (10)$$

As we examine the distribution of normalized wealth, $\langle a\rangle = 1$ is correctly implied. The parameter x can be determined self-consistently and the width of the distribution becomes

$$\langle a^2\rangle = \frac{1-\frac{J}{2N}}{1-\frac{\sigma_m^2}{2J}-\frac{J}{2N}-\frac{\sigma_J^2}{2JN}}. \qquad (11)$$

The numerical data shown in Fig. 1 can be fairly described, especially for the large wealths. The analytical results underestimate the portion of $a<0$. However, the thermodynamic limit is correctly reproduced. In the limit $N\to\infty$, Eq. (10) reduces to Eq. (4) as expected. For a finite N, more people are in debt when σ_J increases. As shown by the analytical expression of Eq. (10), both the large debts and the large wealths are described by power law distributions of the same exponent. The fluctuations in the trading rates will accelerate the wealth accumulation and also the bankruptcy as well.

III. MEAN-FIELD MODEL

We note that the most effective fluctuations in Eq. (7) come from the term $(\Sigma_j J_{ij}\, a_j)/N$. To emphasize such fluctuations, the model is modified as follows:

$$\frac{da_i}{dt} = \left(\eta_i - m - \frac{\sigma_m^2}{2}\right)a_i + J(\zeta_i - a_i), \qquad (12)$$

where the random variable ζ_i is again Gaussian with mean 1 and variance σ. The variables a_i are separate in this mean-field model, which is then specified by four parameters (m, σ_m, J, σ). The fluctuations of redistribution in these two models, Eqs. (7) and (12), are controlled by parameters σ_J and σ, respectively. With naive expectation, these two parameters can be related as $\sigma_J \sim \sigma J\sqrt{N}$. In the thermodynamic limit $N\to\infty$, the fluctuations prescribed by a finite $\sigma>0$ are equivalent to the effects of a divergent $\sigma_J\to\infty$. The numerical results are shown in Fig. 2. Basically, the same features observed only by a finite system in the last section are now preserved in the thermodynamic limit.

The modified model provides yet another advantage: the exact solution. The Fokker-Planck equation for the evolution of the probability density can be written as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial a}\left\{\left[\frac{\sigma_m^2}{2}a - J(1-a)\right]P\right\} + \frac{\sigma_m^2}{2}\frac{\partial}{\partial a}\left[a\frac{\partial}{\partial a}(aP)\right] + \frac{J^2\sigma^2}{2}\frac{\partial^2 P}{\partial a^2}. \qquad (13)$$

After some calculations, the equilibrium solution of the wealth distribution becomes

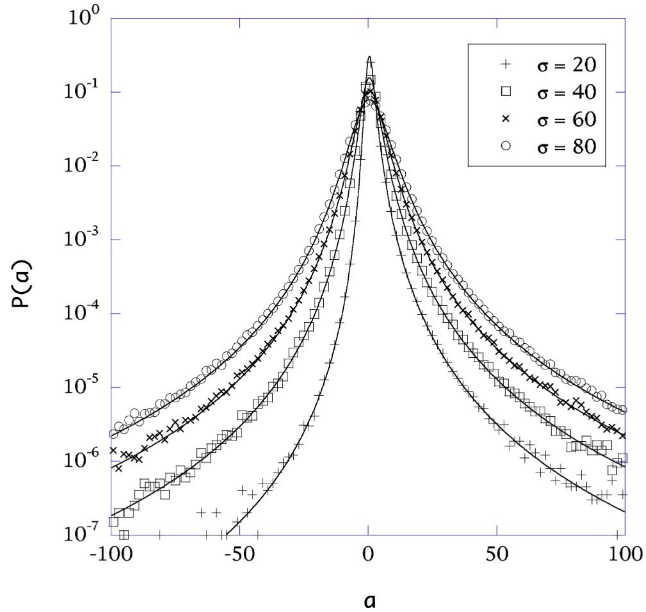

FIG. 2. Wealth distribution in the model of Eq. (12) for various values of σ. The parameters are $(m,\sigma_m,J)=(-0.005,0.1,0.01)$ and $N=10\,000$. The numerical data are averaged over 10 000 simulations at $t=1000$. The solid lines show the analytical results of Eq. (14).

$$P(a) \propto \frac{\exp\left[\frac{2\mu}{\nu}\tan^{-1}\left(\frac{a}{\nu}\right)\right]}{\left(\frac{a^2}{\nu^2}+1\right)^{1+\mu}}, \qquad (14)$$

where the range $\nu = J\sigma/\sigma_m$ and the exponent $\mu = J/\sigma_m^2$. Again, $\langle a\rangle = 1$ is implied and $\langle a^2\rangle$ has the following expression:

$$\langle a^2\rangle = \frac{2+J\sigma^2}{2-\frac{\sigma_m^2}{J}} = \frac{2\mu+\nu^2}{2\mu-1}. \qquad (15)$$

The numerical data can be well reproduced, see Fig. 2.

The same power law distribution for the large wealths, shown in Eq. (5), is preserved. The exponent is determined by parameters σ_m and J, and independent of σ and m. As the fluctuation σ increases, the power law distribution has the same exponent, but the normalization is increased. Without fluctuations ($\sigma=0$), the equilibrium distribution is confined to the range $a>0$; with fluctuations ($\sigma>0$), the wealth distribution is extended to $a<0$. Both the large debts and the large wealths can be described by power law distributions with the same exponent. There are two scales of fluctuations: σ_m and σ. In the limit $\sigma\to 0$, Eq. (14) reduces to Eq. (4) correctly. In the limit $\sigma_m\to 0$, the distribution becomes a Gaussian as expected,

$$P(a) = \sqrt{\frac{1}{\pi J\sigma^2}}\exp\left[-\frac{(a-1)^2}{J\sigma^2}\right]. \qquad (16)$$

IV. DISCUSSIONS

In this paper, we study the effects of redistribution in the processes of wealth accumulation. The emergence of large wealth follows a power law distribution with the exponent determined by the strength of redistribution. With deterministic redistribution, the distribution of small wealth drops strictly to zero as the personal wealth w approaches zero, i.e., the wealth distribution is confined to the positive wealth $w>0$. With random redistribution, the distribution is extended to the negative wealth $w<0$. The width of the distribution is broadened. The extreme wealths, both $w>0$ and $w<0$, are enhanced. The emergence of large debt also follows a power law distribution. The distributions of extreme wealths share the same exponent, which is determined by the average strength of redistribution.

We propose a mean-field model which preserves the fluctuations in the thermodynamic limit and can be solved exactly. The shape of the stationary wealth distribution is the result of two competing mechanisms: the growth and the redistribution. The multiplicative noise in the spontaneous growth broadens the distribution, and thus spreads the wealth unevenly; while the redistribution narrows the distribution and suppresses the difference in personal wealth. In the simulations, the total wealth spreads evenly in the initial configuration, i.e., $P(a)=\delta(a-1)$. All the individuals are equivalent and subjected to the same interactions. The wealth accumulation is the same random process for everyone. The wealth distribution shown in Fig. 2 can also be applied to the temporal fluctuations of personal wealth. Everyone gets the same opportunity to be rich; also everyone gets the same chance to be in debt. As there is no special investment strategy for anyone, those who accumulate a large amount of wealth will soon return to normal. As to those in large debt, the situation will surely improve soon.

In this work, the wealth distribution is taken as the result of simple fluctuations. Compared to realistic operations, we do not allow personal strategy in trading. The saving is not considered. The tax and social welfare are also neglected. The personal connections and network structure are not addressed. However, the simplicity of the model makes it a basic framework to further explore all these complicated effects. For example, the amount of money in each trade can be further restricted. With naive intuition, the wealth accumulation will be slow down and the large wealth could be suppressed. The above analysis, however, shows that the trading redistributes the wealth, and thus suppresses the large wealths. When the trading is restricted, the appearance of large wealths enhances, see Fig. 3. As expected, the large debts enhance as well. In contrast, the large wealths can be suppressed if the spontaneous growth is further restricted, see Fig. 3.

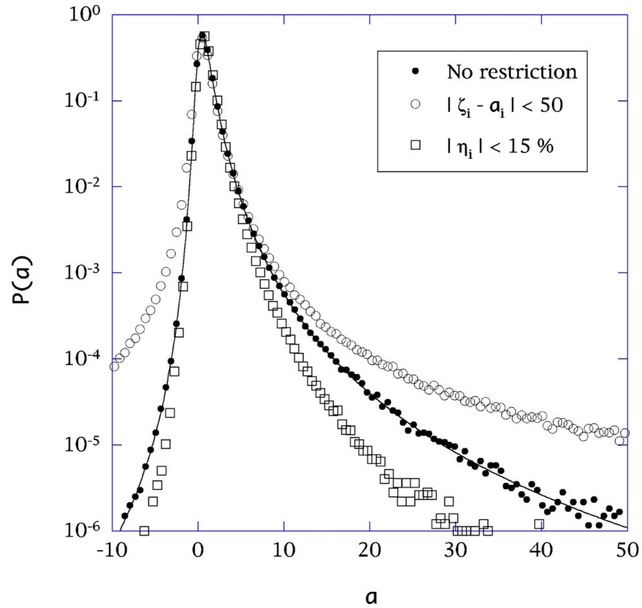

FIG. 3. Wealth distribution in the model of Eq. (12) with various restrictions on the random fluctuations. The parameters are $(m,\sigma_m,J,\sigma)=(-0.005,0.1,0.01,8.8)$ and $N=10\,000$. The value of σ is chosen to have 10% people in debt when no further restrictions are imposed; the solid lines show the analytical results of Eq. (14).

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